

Deformation quantization for coupled harmonic oscillators on a general noncommutative space

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Abstract

Deformation quantization is a powerful tool to quantize some classical systems especially in noncommutative space. In this work we first show that for a class of special Hamiltonian one can easily find relevant time evolution functions and Wigner functions, which are intrinsic important quantities in the deformation quantization theory. Then based on this observation we investigate a two coupled harmonic oscillators system on the general noncommutative phase space by requiring both spatial and momentum coordinates do not commute each other. We derive all the Wigner functions and the corresponding energy spectra for this system, and consider several interesting special cases, which lead to some significant results.

Keywords: Noncommutative space; deformation quantization; star-product; Wigner function; coupled harmonic oscillators.

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1 Introduction

Deformation quantization [1], based on Wigner's quasi-distribution function [2] and Weyl's correspondence between quantum operators and ordinary c -number phase-space functions [3]-[5], have been extensively studied (for recent reviews, see e.g. Ref. [6]-[9]).

In recent years, there has been much interest in the study of physics in noncommutative space. The ideas of noncommutative space-time and field theories defined on such a structure started already in 1947 [10, 11]. It came up again first in the 1980's, when Connes formulated the mathematically rigorous framework of noncommutative geometry [12, 13]. A noncommutative space-time first appeared in physics was in the string theory, namely in the quantization of open string [14], and the noncommutativity of space-time also plays an important role in quantum gravity [15]. Also in condensed matter physics the concept of noncommutative space-time is applied, such as the integer quantum Hall effect [16]. Since the noncommutativity between spatial and time coordinates may lead to some problems with unitarity and causality, usually only spatial noncommutativity is considered. So far quantum theory on the noncommutative space has been studied extensively, and the main approach is based on the Weyl-Moyal correspondence which amounts to replacing the usual

product by a $*$ -product in the noncommutative space. Therefore, deformation quantization has special significance in the study of physical systems on the noncommutative space.

Some works on harmonic oscillators in the noncommutative space from the point of view of deformation quantization have been reported [17]. Considering there are many physical models based on coupled harmonic oscillators [18]-[23], it is interesting to investigate the coupled harmonic oscillators on the noncommutative space. In Ref. [24] two coupled harmonic oscillators on noncommutative plane with space-space noncommutativity were studied, and some results were obtained. In the paper the authors changed the noncommutative problems into problems in the usual commutative space with a well-known coordinates transformation in phase space, and solved the problems with normal quantum mechanics method. But there will be some difficulties of diagonalizing the new Hamiltonian obtained by the coordinates transformation in phase space, and the results such as the Wigner functions are still beyond to the commutative phase space, had not been transformed to those of the noncommutative space. So in the present paper we will remain in noncommutative space and use a new method of deformation quantization to solve this problem.

First we show that for a class of Hamiltonian with special form one can easily derive the relevant $*$ -Exponential function which determines the time evolution of the systems, and all the Wigner functions may be obtained by a Fourier-Dirichlet expansion of the $*$ -Exponential function. When using this observation to the coupled harmonic oscillators on a general noncommutative phase space with both the spatial and momentum coordinates being noncommutative, we derive the explicit form of the relevant $*$ -Exponential function, and then get all the Wigner functions and the corresponding energy spectra of the two coupled harmonic oscillators system on the noncommutative phase space.

This paper is organized as follow. In Sec. 2 we briefly review the fundamental concepts of deformation quantization of a classical system on the noncommutative space. In Sec. 3 we discuss a class of Hamiltonian with special form and present some practical formulas. In Sec. 4 we study the properties of a system consisting of two coupled harmonic oscillators on the noncommutative phase space, derive its energy spectra and all the Wigner functions. In Sec. 5 we discuss some special cases of harmonic oscillators system based on the results in Sec. 4, and compare the results with those we have known in the literature. The summary and concluding remarks are in Sec. 6 .

2 Deformation quantization on noncommutative phase space

Consider a $4D$ general noncommutative phase space, the coordinates of position and momentum are denoted by $\mathbf{x}=\{x_1, x_2\}$ and $\mathbf{p}=\{p_1, p_2\}$, and their corresponding quantum operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ satisfy the following commutation relations [25]

$$[\hat{x}_i, \hat{x}_j] = i\epsilon_{ij}\mu , \quad [\hat{p}_i, \hat{p}_j] = i\epsilon_{ij}\nu , \quad [\hat{x}_i, \hat{p}_j] = i\delta_{ij}\hbar , \quad (1)$$

where $i, j = 1, 2$, and ϵ is the antisymmetric matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In the deformation quantization theory of a classical system in the noncommutative space, one treats (\mathbf{x}, \mathbf{p}) and their functions as classical quantities, but replaces the ordinary

product between these functions by the following generalized $*$ -product [8]

$$\begin{aligned} * &= *_\hbar *_\mu *_\nu \\ &= \exp \left\{ \frac{i\hbar}{2} \left(\overleftarrow{\partial}_{x_i} \overrightarrow{\partial}_{p_i} - \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{x_i} \right) + \frac{i\mu}{2} \epsilon_{ij} \overleftarrow{\partial}_{x_i} \overrightarrow{\partial}_{x_j} + \frac{i\nu}{2} \epsilon_{ij} \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{p_j} \right\}, \end{aligned} \quad (2)$$

here we have used the Einstein summation convention and also in the latter part of this article without additional indication. The variables x_i, p_i on the noncommutative phase space satisfy the following commutation relations similar to (1)

$$[x_i, x_j]_* = i\epsilon_{ij}\mu, \quad [p_i, p_j]_* = i\epsilon_{ij}\nu, \quad [x_i, p_j]_* = i\delta_{ij}\hbar, \quad (3)$$

where the Moyal bracket is defined as $[f, g]_* = f * g - g * f$.

The time evolution function for a time-independent Hamiltonian H of the system is denoted by the $*$ -Exponential function, which is the solution of the following equation

$$\begin{aligned} i\hbar \frac{d}{dt} \text{Exp} \left(\frac{Ht}{i\hbar} \right) &= H(\mathbf{x}, \mathbf{p}) * \text{Exp} \left(\frac{Ht}{i\hbar} \right) \\ &= H \left(x_i + \frac{i\hbar}{2} \overrightarrow{\partial}_{p_i} + \frac{i\mu}{2} \epsilon_{ik} \overrightarrow{\partial}_{x_k}, p_j - \frac{i\hbar}{2} \overrightarrow{\partial}_{x_j} + \frac{i\nu}{2} \epsilon_{jl} \overrightarrow{\partial}_{p_l} \right) \text{Exp} \left(\frac{Ht}{i\hbar} \right). \end{aligned} \quad (4)$$

where $i, j, k, l = 1, 2$. Eq. (4) corresponds to the time-dependent Schrödinger equation, and the $*$ -Exponential function can be expressed by [8]

$$\text{Exp} \left(\frac{Ht}{i\hbar} \right) := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t}{i\hbar} \right)^n (H*)^n, \quad (5)$$

where

$$(H*)^n = \underbrace{H * H * \cdots * H}_{n \text{ times}}.$$

The generalized $*$ -genvalue equation is

$$H * \mathcal{W}_n = \mathcal{W}_n * H = E_n \mathcal{W}_n, \quad (6)$$

where \mathcal{W} is the Wigner function and E is the corresponding energy eigenvalue of the system. Eq. (6) corresponds to the time-independent Schrödinger equation. The relation between the Wigner functions and the $*$ -Exponential function is:

$$\text{Exp} \left(\frac{Ht}{i\hbar} \right) = \sum_{n=0}^{\infty} e^{-iE_n t/\hbar} \mathcal{W}_n, \quad (7)$$

this expression is also called Fourier-Dirichlet expansion for the time-evolution function.

3 A class of Hamiltonian with special form

Generally in order to obtain the $*$ -Exponential function or the Wigner functions for some systems, one may try to solve the differential equation (4) or (6) which contains five or four variables. In many cases it is very difficult, but when the Hamiltonian is some special form, the problem can be simplified.

Now let us consider a Hamiltonian which can be written as a sum of two perfect square parts

$$H = (\mathbf{a} \cdot \mathbf{x} + \mathbf{b} \cdot \mathbf{p})^2 + (\mathbf{c} \cdot \mathbf{x} + \mathbf{d} \cdot \mathbf{p})^2 , \quad (8)$$

where $\mathbf{a}=\{a_1, a_2\}$, $\mathbf{b}=\{b_1, b_2\}$, $\mathbf{c}=\{c_1, c_2\}$ and $\mathbf{d}=\{d_1, d_2\}$, and the coefficients a_i , b_i , c_i and d_i are arbitrary real constants. The time evolution equation (4) with the Hamiltonian (8) will become

$$\begin{aligned} i\hbar \frac{d}{dt} \text{Exp} \left(\frac{Ht}{i\hbar} \right) &= H * \text{Exp} \left(\frac{Ht}{i\hbar} \right) \\ &= \left[\left(a_i \left(x_i + \frac{i\hbar}{2} \vec{\partial}_{p_i} + \frac{i\mu}{2} \epsilon_{ij} \vec{\partial}_{x_j} \right) + b_i \left(p_i - \frac{i\hbar}{2} \vec{\partial}_{x_i} + \frac{i\nu}{2} \epsilon_{ij} \vec{\partial}_{p_j} \right) \right)^2 \right. \\ &\quad \left. + \left(c_i \left(x_i + \frac{i\hbar}{2} \vec{\partial}_{p_i} + \frac{i\mu}{2} \epsilon_{ij} \vec{\partial}_{x_j} \right) + d_i \left(p_i - \frac{i\hbar}{2} \vec{\partial}_{x_i} + \frac{i\nu}{2} \epsilon_{ij} \vec{\partial}_{p_j} \right) \right)^2 \right] \text{Exp} \left(\frac{Ht}{i\hbar} \right). \end{aligned} \quad (9)$$

After some straightforward algebras we arrive at the following form

$$i\hbar \frac{d}{dt} \text{Exp} \left(\frac{Ht}{i\hbar} \right) = \left(H - k^2 \partial_H - k^2 H \partial_H^2 \right) \text{Exp} \left(\frac{Ht}{i\hbar} \right) , \quad (10)$$

where H is the Hamiltonian (8), and

$$\begin{aligned} k &= (a_1 d_1 + a_2 d_2 - b_1 c_1 - b_2 c_2) \hbar + (a_1 c_2 - a_2 c_1) \mu + (b_1 d_2 - b_2 d_1) \nu \\ &= (\mathbf{a} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{c}) \hbar + (\mathbf{a} \wedge \mathbf{c}) \mu + (\mathbf{b} \wedge \mathbf{d}) \nu . \end{aligned} \quad (11)$$

Analogously, the generalized *-genvalue equation (6) becomes

$$H * \mathcal{W}_n = \mathcal{W}_n * H = \left(H - k^2 \partial_H - k^2 H \partial_H^2 \right) \mathcal{W}_n = E_n \mathcal{W}_n . \quad (12)$$

Obviously, Eqs. (10) and (12) will work for any value of the parameters μ and ν , especially for $\mu = 0$ and $\nu = 0$, which correspond to the ordinary quantum theory.

Now the Eqs. (10) and (12) are much simpler than the original form (4) and (6), they only contain one variable H in the right hand side of the equations. The solution of Eq. (10) is [8] :

$$\text{Exp} \left(\frac{Ht}{i\hbar} \right) = \frac{1}{\cos(kt/\hbar)} \exp \left(\frac{H}{ik} \tan \frac{kt}{\hbar} \right) . \quad (13)$$

Using the generating function for the Laguerre polynomials

$$\frac{1}{1+s} \exp \left(\frac{zs}{1+s} \right) = \sum_{n=0}^{\infty} s^n (-1)^n L_n(z) , \quad (14)$$

where $L_n(z)$ is the Laguerre polynomial, with $s = e^{-2ikt/\hbar}$ and $z = 2H/k$, one can write the *-Exponential function (13) as

$$\text{Exp} \left(\frac{Ht}{i\hbar} \right) = \sum_{n=0}^{\infty} 2(-1)^n e^{-ikt(2n+1)/\hbar} e^{-H/k} L_n \left(\frac{2H}{k} \right) . \quad (15)$$

Comparing this expression with the Fourier-Dirichlet expansion for the time evolution function (7), we get the Wigner functions and the corresponding energy eigenvalues¹:

$$\mathcal{W}_n = e^{-H/k} L_n \left(\frac{2H}{k} \right) , \quad E_n = (2n+1)k . \quad (16)$$

They are also the solutions of Eq. (12).

Obviously, for most physical systems their Hamiltonian functions are not exactly the form (8), but we can use it to simplify our calculation in some cases, for example in the case of coupled harmonic oscillators.

4 Coupled harmonic oscillators on noncommutative space

Now consider the two coupled harmonic oscillators system [21, 24] on the phase space, the Hamiltonian can be written as

$$H_0 = \frac{1}{2m_1} P_1^2 + \frac{1}{2m_2} P_2^2 + \frac{1}{2} (C_1 X_1^2 + C_2 X_2^2 + C_3 X_1 X_2) , \quad (17)$$

where m_1, m_2 are masses, and C_1, C_2, C_3 are constant parameters. After rescaling the coordinates of the phase space

$$\begin{aligned} x_1 &= \left(\frac{m_1}{m_2} \right)^{\frac{1}{4}} X_1, & x_2 &= \left(\frac{m_2}{m_1} \right)^{\frac{1}{4}} X_2, \\ p_1 &= \left(\frac{m_2}{m_1} \right)^{\frac{1}{4}} P_1, & p_2 &= \left(\frac{m_1}{m_2} \right)^{\frac{1}{4}} P_2, \end{aligned} \quad (18)$$

one can rewrite H_0 as

$$H_1 = \frac{1}{2m} (p_1^2 + p_2^2) + \frac{1}{2} (c_1 x_1^2 + c_2 x_2^2 + c_3 x_1 x_2) , \quad (19)$$

where m, c_1, c_2 and c_3 are

$$m = \sqrt{m_1 m_2}, \quad c_1 = C_1 \sqrt{\frac{m_2}{m_1}}, \quad c_2 = C_2 \sqrt{\frac{m_1}{m_2}}, \quad c_3 = C_3 . \quad (20)$$

To remove the interaction term in H_1 (19), one may use the following transformation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos \frac{\alpha}{2} & -\sin \frac{\alpha}{2} \\ \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cos \frac{\alpha}{2} & -\sin \frac{\alpha}{2} \\ \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad (21)$$

which is a unitary rotation with the mixing angle α on the phase space. Applying the transformation (21) to (19), and letting the parameter α satisfy the condition

$$\tan \alpha = \frac{c_3}{c_2 - c_1} , \quad (22)$$

one can find the Hamiltonian (19) will be

$$H_2 = \frac{1}{2m} (q_1^2 + q_2^2) + \frac{K}{2} (e^{2\eta} y_1^2 + e^{-2\eta} y_2^2) , \quad (23)$$

¹In this work we will always ignore the normalized constant of any Wigner functions.

where

$$K = \frac{1}{2} \sqrt{4c_1c_2 - c_3^2}, \quad e^{2\eta} = \frac{c_1 + c_2 + \sqrt{(c_1 - c_2)^2 + c_3^2}}{\sqrt{4c_1c_2 - c_3^2}}, \quad (24)$$

and the condition $4c_1c_2 > c_3^2$ must be fulfilled. Since the transformation (21) does not change the structure of the generalized \ast -product (2), the coordinates y_1, y_2, q_1 and q_2 should satisfy the commutation relations (3) the same as those of X_1, X_2, P_1 and P_2 .

The Hamiltonian H_2 (23) can be separated into the following two parts

$$\mathcal{H}_1 = \left(\frac{e^\eta \sqrt{K} \sin a}{\sqrt{2}} y_1 + \frac{\cos a}{\sqrt{2m}} q_2 \right)^2 + \left(\frac{e^{-\eta} \sqrt{K} \sin b}{\sqrt{2}} y_2 + \frac{\cos b}{\sqrt{2m}} q_1 \right)^2, \quad (25)$$

$$\mathcal{H}_2 = \left(\frac{e^\eta \sqrt{K} \cos a}{\sqrt{2}} y_1 - \frac{\sin a}{\sqrt{2m}} q_2 \right)^2 + \left(\frac{e^{-\eta} \sqrt{K} \cos b}{\sqrt{2}} y_2 - \frac{\sin b}{\sqrt{2m}} q_1 \right)^2. \quad (26)$$

Obviously, with any value of a and b , there should be $\mathcal{H}_1 + \mathcal{H}_2 = H_2$.

Using the observation obtained in Sec. 3, we will immediately get the expressions of the time-evolution functions for \mathcal{H}_1 (25) and \mathcal{H}_2 (26)

$$\text{Exp}_1 \left(\frac{\mathcal{H}_1 t}{i\hbar} \right) = \frac{1}{\cos(k_1 t/\hbar)} \exp \left(\frac{\mathcal{H}_1}{ik_1} \tan \frac{k_1 t}{\hbar} \right), \quad (27)$$

$$\text{Exp}_2 \left(\frac{\mathcal{H}_2 t}{i\hbar} \right) = \frac{1}{\cos(k_2 t/\hbar)} \exp \left(\frac{\mathcal{H}_2}{ik_2} \tan \frac{k_2 t}{\hbar} \right), \quad (28)$$

where

$$k_1 = \frac{\hbar \sqrt{K}}{2\sqrt{m}} (e^\eta \sin a \cos b - e^{-\eta} \sin b \cos a) + \frac{K\mu}{2} \sin a \sin b - \frac{\nu}{2m} \cos a \cos b, \quad (29)$$

$$k_2 = \frac{\hbar \sqrt{K}}{2\sqrt{m}} (e^{-\eta} \sin a \cos b - e^\eta \sin b \cos a) + \frac{K\mu}{2} \cos a \cos b - \frac{\nu}{2m} \sin a \sin b.$$

The Wigner functions and the corresponding energy spectra are

$$\mathcal{W}_{n_1}^{(1)} = e^{-\mathcal{H}_1/k_1} L_{n_1} \left(\frac{2\mathcal{H}_1}{k_1} \right), \quad E_{n_1}^{(1)} = (2n_1 + 1)k_1; \quad (30)$$

$$\mathcal{W}_{n_2}^{(2)} = e^{-\mathcal{H}_2/k_2} L_{n_2} \left(\frac{2\mathcal{H}_2}{k_2} \right), \quad E_{n_2}^{(2)} = (2n_2 + 1)k_2. \quad (31)$$

It is easy to verify when a and b take the following values

$$\sin(a - b) = \frac{\hbar \sqrt{Km}(e^\eta + e^{-\eta})}{\beta_1}, \quad \cos(a - b) = \frac{Km\mu - \nu}{\beta_1}, \quad (32)$$

$$\sin(a + b) = \frac{\hbar \sqrt{Km}(e^\eta - e^{-\eta})}{\beta_2}, \quad \cos(a + b) = -\frac{Km\mu + \nu}{\beta_2},$$

or

$$a = \frac{1}{2} \left(\arctan \frac{\hbar \sqrt{Km}(e^\eta - e^{-\eta})}{-(Km\mu + \nu)} + \arctan \frac{\hbar \sqrt{Km}(e^\eta + e^{-\eta})}{Km\mu - \nu} \right), \quad (33)$$

$$b = \frac{1}{2} \left(\arctan \frac{\hbar \sqrt{Km}(e^\eta - e^{-\eta})}{-(Km\mu + \nu)} - \arctan \frac{\hbar \sqrt{Km}(e^\eta + e^{-\eta})}{Km\mu - \nu} \right),$$

then \mathcal{H}_1 (25) will be commutative with \mathcal{H}_2 (26) under the Moyal bracket

$$[\mathcal{H}_1, \mathcal{H}_2]_* = \mathcal{H}_1 * \mathcal{H}_2 - \mathcal{H}_2 * \mathcal{H}_1 = 0, \quad (34)$$

and their generalized $*$ -product is equal to their ordinary product

$$\mathcal{H}_1 * \mathcal{H}_2 = \mathcal{H}_1 \mathcal{H}_2 = \mathcal{H}_2 * \mathcal{H}_1. \quad (35)$$

β_1 and β_2 in (32) are

$$\begin{aligned} \beta_1 &= \sqrt{(e^\eta + e^{-\eta})^2 \hbar^2 K m + (K m \mu - \nu)^2}, \\ \beta_2 &= \sqrt{(e^\eta - e^{-\eta})^2 \hbar^2 K m + (K m \mu + \nu)^2}, \end{aligned} \quad (36)$$

or

$$\beta_1 = (e^\eta + e^{-\eta}) \hbar \sqrt{K m} \sqrt{1 + \Delta_1}, \quad \beta_2 = (e^\eta - e^{-\eta}) \hbar \sqrt{K m} \sqrt{1 + \Delta_2}, \quad (37)$$

with

$$\Delta_1 = \frac{(K m \mu - \nu)^2}{(e^\eta + e^{-\eta})^2 \hbar^2 K m}, \quad \Delta_2 = \frac{(K m \mu + \nu)^2}{(e^\eta - e^{-\eta})^2 \hbar^2 K m}, \quad (38)$$

here Δ_1 and Δ_2 denote the effect of the noncommutativity of the phase space. When $\mu = 0$ and $\nu = 0$, $\Delta_1 = \Delta_2 = 0$, it returns to the ordinary commutative phase space.

With (29) and (32), k_1 and k_2 can be written as

$$\begin{aligned} k_1 &= \frac{1}{4m}(\beta_1 + \beta_2) = \frac{\hbar\omega}{4} \left((e^\eta + e^{-\eta}) \sqrt{1 + \Delta_1} + (e^\eta - e^{-\eta}) \sqrt{1 + \Delta_2} \right), \\ k_2 &= \frac{1}{4m}(\beta_1 - \beta_2) = \frac{\hbar\omega}{4} \left((e^\eta + e^{-\eta}) \sqrt{1 + \Delta_1} - (e^\eta - e^{-\eta}) \sqrt{1 + \Delta_2} \right), \end{aligned} \quad (39)$$

where $\omega = \sqrt{K/m}$. Then from the definition of the $*$ -Exponential function (5) and Eq. (35), we obtain the time evolution function for the coupled harmonic oscillators H_2 (23) on the noncommutative phase space

$$\begin{aligned} \text{Exp}_{H_2} \left(\frac{H_2 t}{i\hbar} \right) &= \text{Exp}_1 \left(\frac{\mathcal{H}_1 t}{i\hbar} \right) * \text{Exp}_2 \left(\frac{\mathcal{H}_2 t}{i\hbar} \right) \\ &= \text{Exp}_1 \left(\frac{\mathcal{H}_1 t}{i\hbar} \right) \text{Exp}_2 \left(\frac{\mathcal{H}_2 t}{i\hbar} \right) \\ &= \frac{1}{\cos(k_1 t/\hbar)} \exp \left(\frac{\mathcal{H}_1}{ik_1} \tan \frac{k_1 t}{\hbar} \right) \frac{1}{\cos(k_2 t/\hbar)} \exp \left(\frac{\mathcal{H}_2}{ik_2} \tan \frac{k_2 t}{\hbar} \right), \end{aligned} \quad (40)$$

and the Wigner functions are

$$\begin{aligned} \mathcal{W}_{n_1 n_2} &= \mathcal{W}_{n_1}^{(1)} * \mathcal{W}_{n_2}^{(2)} = \mathcal{W}_{n_1}^{(1)} \mathcal{W}_{n_2}^{(2)} \\ &= e^{-\mathcal{H}_1/k_1 - \mathcal{H}_2/k_2} L_{n_1} \left(\frac{2\mathcal{H}_1}{k_1} \right) L_{n_2} \left(\frac{2\mathcal{H}_2}{k_2} \right), \end{aligned} \quad (41)$$

the corresponding energies are

$$\begin{aligned} E_{n_1 n_2} &= E_{n_1}^{(1)} + E_{n_2}^{(2)} = (2n_1 + 1)k_1 + (2n_2 + 1)k_2 \\ &= \frac{1}{2m} \left((n_1 + n_2 + 1)\beta_1 + (n_1 - n_2)\beta_2 \right) \\ &= \frac{\hbar\omega}{2} \left((n_1 + n_2 + 1)(e^\eta + e^{-\eta}) \sqrt{1 + \Delta_1} + (n_1 - n_2)(e^\eta - e^{-\eta}) \sqrt{1 + \Delta_2} \right). \end{aligned} \quad (42)$$

Through the inverse transform of (21), all the results above can be easily expressed in terms of the original variables of the noncommutative phase space (x_1, x_2, p_1, p_2) or (X_1, X_2, P_1, P_2) . So, based on the observation in Sec. 3, we derive a satisfactory result of the two coupled harmonic oscillators system on the noncommutative phase space with deformation quantization method, obtain all the Wigner functions and the energy spectra.

5 Some particular cases

5.1 $\mu = 0$ and $\nu = 0$

If $\mu = \nu = 0$, then $\Delta_1 = \Delta_2 = 0$, the noncommutative phase space will reduce to the ordinary phase space. In this case, one has

$$\beta_1 = (e^\eta + e^{-\eta})\hbar\sqrt{Km}, \quad \beta_2 = (e^\eta - e^{-\eta})\hbar\sqrt{Km}, \quad (43)$$

and

$$k_1 = \frac{\hbar\sqrt{K}}{2\sqrt{m}}e^\eta = \frac{\hbar\omega}{2}e^\eta, \quad k_2 = \frac{\hbar\sqrt{K}}{2\sqrt{m}}e^{-\eta} = \frac{\hbar\omega}{2}e^{-\eta}. \quad (44)$$

From (32) we have

$$\sin(a-b) = 1, \quad \cos(a-b) = 0, \quad \sin(a+b) = 1, \quad \cos(a+b) = 0, \quad (45)$$

so we may choose $a = \pi/2$ and $b = 0$. Then \mathcal{H}_1 (25) and \mathcal{H}_2 (26) become

$$\mathcal{H}_1 = \frac{1}{2m}q_1^2 + \frac{K}{2}e^{2\eta}y_1^2, \quad \mathcal{H}_2 = \frac{1}{2m}q_2^2 + \frac{K}{2}e^{-2\eta}y_2^2, \quad (46)$$

and the Wigner functions are

$$\begin{aligned} \mathcal{W}_{n_1 n_2} &= e^{-\mathcal{H}_1/k_1 - \mathcal{H}_2/k_2} L_{n_1} \left(\frac{2\mathcal{H}_1}{k_1} \right) L_{n_2} \left(\frac{2\mathcal{H}_2}{k_2} \right) \\ &= \exp \left\{ -\frac{1}{\hbar\sqrt{Km}} (e^{-\eta}q_1^2 + e^\eta q_2^2) - \frac{\sqrt{Km}}{\hbar} (e^\eta y_1^2 + e^{-\eta} y_2^2) \right\} \\ &\quad \times L_{n_1} \left(\frac{2}{\hbar\sqrt{Km}} e^{-\eta} q_1^2 + \frac{2\sqrt{Km}}{\hbar} e^\eta y_1^2 \right) \\ &\quad \times L_{n_2} \left(\frac{2}{\hbar\sqrt{Km}} e^\eta q_2^2 + \frac{2\sqrt{Km}}{\hbar} e^{-\eta} y_2^2 \right), \end{aligned} \quad (47)$$

with the corresponding energy spectra

$$\begin{aligned} E_{n_1 n_2} &= (2n_1 + 1)k_1 + (2n_2 + 1)k_2 = \frac{\hbar\sqrt{K}}{2\sqrt{m}} (e^\eta(2n_1 + 1) + e^{-\eta}(2n_2 + 1)) \\ &= \hbar\omega \left(e^\eta \left(n_1 + \frac{1}{2} \right) + e^{-\eta} \left(n_2 + \frac{1}{2} \right) \right). \end{aligned} \quad (48)$$

These results are exactly the same as those in Ref. [24].

5.2 $\mu = -\nu = \theta$

For simplicity, we also choose $e^\eta = K = m = 1$ (e.g. let $c_1=c_2=1$ and $c_3=0$ in (24)), then β_1, β_2 and k_1, k_2 become

$$\begin{aligned}\beta_1 &= 2\sqrt{\hbar^2 + \theta^2}, \quad \beta_2 = 0; \\ k_1 &= k_2 = \frac{\sqrt{\hbar^2 + \theta^2}}{2},\end{aligned}\tag{49}$$

and

$$\sin(a-b) = \frac{\hbar}{\sqrt{\hbar^2 + \theta^2}}, \quad \cos(a-b) = \frac{\theta}{\sqrt{\hbar^2 + \theta^2}},\tag{50}$$

but the value $(a+b)$ can not be fixed, so we may choose $a = \pi$ for simplicity, then we have

$$\mathcal{H}_1 = \frac{1}{2} \left(\frac{(\hbar y_2 - \theta q_1)^2}{\hbar^2 + \theta^2} + q_2^2 \right), \quad \mathcal{H}_2 = \frac{1}{2} \left(y_1^2 + \frac{(\hbar q_1 + \theta y_2)^2}{\hbar^2 + \theta^2} \right).\tag{51}$$

Thus the Wigner functions are

$$\begin{aligned}\mathcal{W}_{n_1 n_2} &= e^{-\mathcal{H}_1/k_1 - \mathcal{H}_2/k_2} L_{n_1} \left(\frac{2\mathcal{H}_1}{k_1} \right) L_{n_2} \left(\frac{2\mathcal{H}_2}{k_2} \right) \\ &= \exp \left\{ -\frac{1}{\sqrt{\hbar^2 + \theta^2}} \left(y_1^2 + \frac{(\hbar q_1 + \theta y_2)^2}{\hbar^2 + \theta^2} + \frac{(\hbar y_2 - \theta q_1)^2}{\hbar^2 + \theta^2} + q_2^2 \right) \right\} \\ &\quad \times L_{n_1} \left[\frac{2}{\sqrt{\hbar^2 + \theta^2}} \left(\frac{(\hbar y_2 - \theta q_1)^2}{\hbar^2 + \theta^2} + q_2^2 \right) \right] \\ &\quad \times L_{n_2} \left[\frac{2}{\sqrt{\hbar^2 + \theta^2}} \left(y_1^2 + \frac{(\hbar q_1 + \theta y_2)^2}{\hbar^2 + \theta^2} \right) \right],\end{aligned}\tag{52}$$

and the corresponding energies are

$$\begin{aligned}E_{n_1 n_2} &= \frac{1}{2} \left((n_1 + n_2 + 1)\beta_1 + (n_1 - n_2)\beta_2 \right) \\ &= (n_1 + n_2 + 1)\sqrt{\hbar^2 + \theta^2}.\end{aligned}\tag{53}$$

Therefore, in the case of $\mu = -\nu = \theta$, (41) and (42) will reduce to the results of Ref. [17].

5.3 $\mu, \nu \ll \hbar$

When $\mu, \nu \ll \hbar$, it leads to $\Delta_1, \Delta_2 \ll 1$, and

$$\sqrt{1 + \Delta_1} \approx 1 + \frac{\Delta_1}{2}, \quad \sqrt{1 + \Delta_2} \approx 1 + \frac{\Delta_2}{2}.\tag{54}$$

Thus the energy (42) becomes

$$\begin{aligned}E_{n_1 n_2} &= \frac{\hbar\omega}{2} \left((n_1 + n_2 + 1)(e^\eta + e^{-\eta})\sqrt{1 + \Delta_1} + (n_1 - n_2)(e^\eta - e^{-\eta})\sqrt{1 + \Delta_2} \right) \\ &\approx \hbar\omega \left[e^\eta \left(n_1 + \frac{1}{2} \right) + e^{-\eta} \left(n_2 + \frac{1}{2} \right) + \right. \\ &\quad \left. + \frac{n_1 + n_2 + 1}{4}(e^\eta + e^{-\eta})\Delta_1 + \frac{n_1 - n_2}{4}(e^\eta - e^{-\eta})\Delta_2 \right].\end{aligned}\tag{55}$$

Comparing this result with the energy spectrum (48) on the commutative phase space, we find that, if the noncommutativity parameters μ and ν are much smaller than the Planck constant, the noncommutativity of the space structure will cause a shift of the energy spectrum of the coupled harmonic oscillator system. So if we find a deviation of the spectra of the coupled harmonic oscillators system from the standard ones in some experiments, we may measure and determine the values of the noncommutativity parameters μ and ν and investigate the effects of the noncommutativity.

6 Conclusion

In this paper, we first consider deformation quantization for a class of systems with special Hamiltonian. Using this method, we investigate the two coupled harmonic oscillators system on a general noncommutative phase space, and obtain the explicit expression of the time-evolution function for this system, and furthermore, derive all the Wigner functions and the corresponding energies. Since the study is for a very general noncommutative space, and the results are also rigorous and exact, so the results certainly can be reduced to specific expressions reported in the literature in different specific cases. In particular, the results show that when the spatial or the momentum coordinates do not commute with each other, the energy spectra of the coupled harmonic oscillators system will have a shift comparing to the standard ones, and this phenomenon indicates a possible way to explore the noncommutativity of the spatial or the momentum coordinates in future experiments. When both the spatial and the momentum coordinates are commutative with each other, the results return to those of the ordinary case. We also discuss a case in which the noncommutativity parameters μ and ν are much smaller than the Planck constant, and calculate the quantity of the energy shift, so if we could compare the quantity with some relevant experimental data, we would get the bound of the noncommutativity parameters, which will be very significant.

All these results come from our observation in Sec. 3, and we believe that this method should be useful in some other cases.

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